Role of Compaction Ratio in the Mathematical Model of Progressive Collapse

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Abstract

We derive a mathematical model of progressive collapse and examine role of compaction. Contrary to a previous result by Bažant and Verdure, J. Engr. Mech. ASCE 133 (2006) 308, we find that compaction slows down the avalanche by effectively increasing the resistive force. We compare currently available estimates of the resistive force, that of Bažant and Verdure (2006) corrected for compaction for World Trade Center (WTC) 2, and of Beck, www.arxiv.org:physics/0609105, for WTC 1 and 2. We concentrate on a damage wave propagating through the building before the avalanche that figures in both models: an implicit heat wave that reduces the resistive force of the building by 60% in Bažant and Verdure (2006), or a wave of massive destruction that reduces the resistive force by 75% in Beck (2006). We show that the avalanche cannot supply the energy to the heat wave as this increases the resistive force by two orders of magnitude. We thus reaffirm the conclusion of Beck (2006) that the avalanche is initiated in the wake of the damage wave.

Bažant and Verdure[1] proposed the following mathematical model to describe the progressive collapse in a tall building of a homogeneous longitudinal density ρ_0,

$$\rho_0 \left(1 - \kappa\right) \frac{d}{dt} \left(x_2 \, \dot{x}_2\right) = R(x_2) + \rho_0 \, g \, x_2, \tag{1}$$

where x_2 is the position of the avalanche front, $\rho_0 = M/H$ with M the total mass and H the total height of the building and g is the gravity, while $R = R(x_2)$ is a local resistive force. Here, the dot above the quantity indicates its differentiation with respect to time. The compaction ratio κ is defined as

$$\kappa = \frac{\rho_0}{\rho}, \tag{2}$$

where ρ is a density of the "compacted" section of the building, cf. Fig. 1.

We reexamine the steps that lead to Eq. (1) from the point of view of classical mechanics. As can be seen from Fig. 1 we can choose between two generalized coordinates. The first choice is $x_1 = x_1(t)$ which for $t > 0$ well describes the motion of the avalanche. The second choice is $x_2 = x_2(t)$, which represents the position of the avalanche front - an idealized point-like boundary between the stationary and the moving part of the building, at which the compaction takes place. Motion of the avalanche front is more complex than the motion of the avalanche as it combines the motion of the avalanche with its spatial growth due to non-zero compaction ratio. While care must be exercised when deriving an equation of motion for each of them, the final result may not depend on the choice of generalized coordinate.

For simplicity, we assume that the total energy and the total mass in the system building-avalanche is conserved. We note that in that case we obtain the fastest avalanche, as then there are no conversion losses of the potential energy of the building into the kinetic and then into the crushing energy of the avalanche. Also, the conservation of energy allows us to use Lagrangian formalism to derive the equation of motion.

First, we state the two constraints of descent,

$$x_1 - x_0 = h, \tag{3a}$$

$$\rho_0 \, H = \rho_0 \left(x_1 - x_0\right) + \rho \left(x_2 - x_1\right) + \rho_0 \left(H - x_2\right). \tag{3b}$$

Here, the top section stretches from $x_0 = x_0(t)$ down to $x_1 = x_1(t)$, while the compacted building occupies from x_1 down to $x_2 = x_2(t)$. The point x_2 is the avalanche front. With

Eq. (3a) we state that the length of the top section h does not change in descent, while with Eq. (3b) we express the conservation of the mass of the building. Differentiation of Eq. (3) with respect to time yields dynamical constraints, $\dot{x}_1 = \dot{x}_0$ and $\dot{x}_1 = (1 - \kappa)\,\dot{x}_2$.

Second, we find kinetic, potential and latent energy necessary for the Lagrangian formulation. The kinetic energy K is given by $K = \frac{1}{2}\int dx\,\rho(x)\,v^2(x)$, where the velocity distribution in the avalanche is $v(x) = \dot{x}_1$ for $x \in [x_0, x_2\rangle$. The kinetic energy of the avalanche is thus

$$K = \frac{1}{2}\rho_0\,x_2\,\dot{x}_1^2 = \frac{1}{2}\rho_0(1 - \kappa)^2\,x_2\,\dot{x}_2^2. \tag{4}$$

The potential energy U of the whole building is $U = -\int dx\,\rho(x)\,x\,g$, yielding

$$U = -\frac{1}{2}\,g\,\rho_0\,\left(H^2 + (1 - \kappa)\,(x_2^2 - h^2)\right). \tag{5}$$

The latent energy $L = L(x_2) = -\int_H^{x_2} dx'\,R(x')$, produces the resistive force of the building, $R(x_2) = -\partial L(x_2)/\partial x_2$.

Given a Lagrangian $\mathcal{L} = K - U - L$, which is a function of a generalized coordinate x and its generalized velocity $\dot{x}$, the equation of motion follows from $\frac{d}{dt}\partial\mathcal{L}/\partial\dot{x} = \partial\mathcal{L}/\partial x$. We recall that we have two choices for the generalized coordinate: $x \equiv x_1$ for the motion of avalanche, or $x \equiv x_2$ for the avalanche front. If x_2 is chosen as a generalized coordinate, we simply obtain,

$$\rho_0\,(1 - \kappa)^2\,\frac{d}{dt}\,(x_2\,\dot{x}_2) = R(x_2) + \rho_0\,g\,(1 - \kappa)\,x_2 + \frac{\eta\,\rho_0}{2}\,(1 - \kappa)^2\,\dot{x}_2^2. \tag{6}$$

With x_1 as a generalized coordinate we note that $\partial/\partial x_1 = (1 - \kappa)^{-1}\partial/\partial x_2$, yielding

$$\rho_0\,(1 - \kappa)\,\frac{d}{dt}\,(x_2\,\dot{x}_2) = \frac{R(x_2)}{1 - \kappa} + \rho_0\,g\,x_2 + \frac{\eta\,\rho_0}{2}\,(1 - \kappa)\,\dot{x}_2^2. \tag{7}$$

As expected, the two equations of motion are identical. In Eqs. (6) and (7) we introduced an additional parameter η which may take values 1, if the total energy is conserved, or 0, if this is not the case. A distinction between the two cases is discussed in [2]. Comparison between Eq. (7) and Eq. (1) of Bažant and Verdure's shows that due to non-zero compaction the avalanche propagates through the building faster than it travels, which leads to an amplification of the building's resistive force by the factor $(1 - \kappa)^{-1} > 1$. In other words, the avalanche front in Bažant and Verdure's model, Eq. (1), is faster than the one proposed in Eq. (7). Eq. (7) as a proposed correction to Eq. (1) is a major result of this technical note. We next discuss the estimates for the resistive force R that can be found in the literature.

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Bažant and Verdure [1] analyzed collapse of World Trade Center 2 in terms of a model (1). They assumed that R is a constant throughout the building's primary and secondary zone, where $R = \Delta L / \Delta H$. For a crushing energy they made an educated guess, $\Delta L = 2.4$ GNm, with the floor height being $\Delta H = 3.7$ m. Considering the total mass of the building to be $M = 3.2 \cdot 10^8$ kg, this yields for the resistive force $R/(M\,g) = r = 0.2$, as their initial estimate. They noted that in order for the avalanche to reach the ground level in $_2T \simeq 10.8$ s [6] they had to use $R/2$ instead of R. The 50% reduction, they argued, came from heat (p.15, top paragraph of the on-line edition of [1]). That is, an assumption built in their model is that the avalanche pushes a heat wave in front of itself which reduces the strength of the building by 50%. Using their value for compaction $\kappa \simeq 0.2$ and the corrected equation of motion proposed here, requires R to be reduced by 60% instead.

Here the following comment is in place. If the avalanche were supplying energy to the heat wave then R in Eq. (1) splits in two components. First, original R decreases, say, by 60% as discussed earlier. However, an additional resistive force $(R)_{heat}$ appears which describes the rate of transfer of energy from the avalanche to the heat wave per unit length. A simple estimate shows [7] that $(R)_{heat}/(M\,g) \sim 20 \gg R$. It is thus obvious that the 60%-strength-reducing heat wave could not have been created or maintained by the avalanche. On the contrary, if the avalanche were in fact heating the steel, then this acted as a resistive force comparable to R. E.g., the avalanche warming the steel by $\Delta T = 10$ K yields $(R)_{heat}/(M\,g) = r \sim 0.1$ which is comparable to the observed resistive force. We do mention, however, that the authoritative document on collapse of World Trade Center 1 and 2, the NIST report [3] explicitely states that no elevated temperatures were observed in the secondary zone.

Rather than guessing, we proposed a procedure for estimating R [4], where one first finds the ultimate yield strength Y of the vertical columns using their specifications, following which the resistive force R is estimated from a simple linear model, $R = \epsilon \cdot Y$, with $\epsilon = 0.25$ being the ultimate yield strain of structural steel under compression. Applying this to WTC 1 and 2 led to an initial estimate of $R(x_2)/(M\,g) = r + s \cdot (x_2/H)$, with $r \simeq 0.2$ and $s \simeq 0.7$. We analyzed the collapse of World Trade Centers 1 and 2 where we divided the building into the primary and the secondary zone where we allowed R_I in the primary zone to be considerably smaller than R_{II} in the secondary zone, $R_I = 1/4 \cdot R_{II}$ for WTC 1 and $R_I = 3/8 \cdot R_{II}$ for WTC 2, while we neglected the compaction altogether. We did this

subdivision after the statement from the NIST report that the damage to the buildings was concentrated in their primary zones while leaving the secondary zones intact. We found that to reach the collapse time $_2T$ the initial estimate of R had to be reduced by 75%. In our report we dubbed the 75%-strength-reducing wave that preceeded the avalanche the wave of massive destruction (WMD). As the avalanche was not supplying the energy to the WMD, and the WMD propagated before the avalanche, we concluded that the WMD caused the avalanche.

Currently, we can only speculate about the source of the 60-75%-strength-reducing wave and its coupling to the avalanche. However, a piece of information that would provide an important insight is the descent curve, which describes position as a function of time of some visible part of the building, say its top, $x_0 = x_0(t)$. Once the descent curve is known it is the acceleration, $\ddot{x}_0 = \ddot{x}_0(t)$, that can be directly connected to R through a mathematical model. In fact, in [5] we examine the descent curve available for WTC 7 in terms of a corrected model of "crush-up" mode of progressive collapse and identify the phases of descent.

Lastly, as the collapses of World Trade Centers resemble controlled demolition it would be instructive to apply the methods discussed in this and other articles to other buildings that are known to have been destroyed in controlled fashion.

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[1] Z. P. Bažant and M. Verdure, J. Engr. Mech. ASCE **133** (2006) 308.

[2] C. P. Pesce, J. Appl. Mech. ASME **70** (2003) 751.

[3] NIST National Construction Safety Team, NIST NCSTAR 1 - Final Report on the Collapse of the World Trade Center Towers, U. S. Government Printing Office, Washington D.C., 2005.

[4] C. M. Beck, 2006-2007, http://www.arxiv.org, article *physics/0609105*.

[5] C. M. Beck, 2008, http://www.arxiv.org, article *physics/0806.4792*.

[6] $_2T \simeq 10.8$ s is an unofficial estimate of the duration of collapse of World Trade Center 2.

[7] This estimate for the magnitude of the resistive force due to heat dissipation follows from $(R)_{heat} = dW_{heat}/dx_2$, with $W_{heat} = \sigma \left(m(x_2) - m(x_1) \right) C \, \Delta T$, and $m(x) = \rho_0 \, x$ assuming the uniform distribution of mass. Taking the heat capacity of the steel $C \simeq 486$ J/(kg K), $H = 417$ m for the height of the building, $\Delta T = 300$ K for the increase in the steel temperature

at which steel strength is reduced by 50%, and $0.5 \leq \sigma \leq 0.9$ for the ratio of the mass of steel to the total mass of the building. This yields $(R)_{heat}/(M\,g) = \sigma\,C\,\Delta T/(H\,g) = 17.8$, which is by two orders of magnitude greater than $r \sim 0.2$, the apparent resistive force of the buildings.
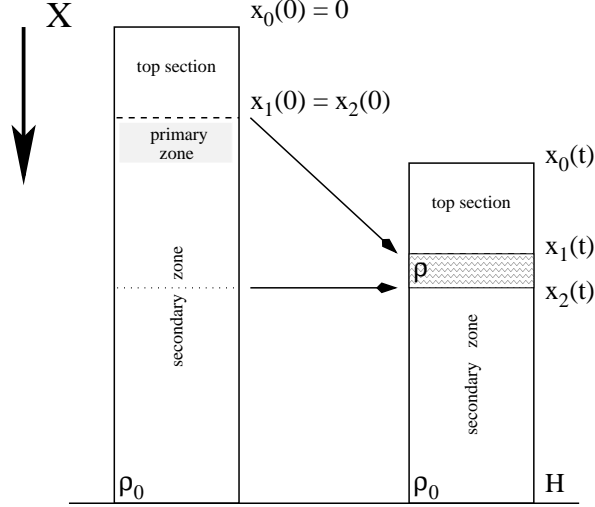
FIGURES

FIG. 1: Propagation of an avalanche in a tall building of uniform density ρ_0. Extraneous factors cause initial weakening of the load bearing structure in the primary zone, leaving the building below (secondary zone) intact. The avalanche forms at the top of the primary zone, which propagates and compacts the building underneath from ρ_0 to ρ, with $\kappa = \rho_0/\rho \ll 1$ being a compaction ratio. The idealized scenario allows us to identify the following points: x_0, the top of the building; x_1, beginning of the compacted section of the avalanche; and x_2, the location of the avalanche front. Here it is implied that it is easier for the avalanche to drop ("crush down") then to stop and compact the top section instead ("crush up").